THEMATIC ARTICLES
ARTYKUŁY TEMATYCZNE

EETP Vol. 16, 2021, No. 3(61)
ISSN 1896-2327 / e-ISSN 2353-7787
DOI: 10.35765/eetp.2021.1661.01

Submitted: 24.04.2021
Accepted: 21.07.2021

Suggested citation: Żądło-Treder J. (2021). *Iconic and symbolic representation in early mathematics teaching*, "Elementary Education in Theory and Practice", vol. 16, no. 3(61), pp. 11-25. DOI: 10.35765/eetp.2021.1661.01

Joanna Żądło-Treder
ORCID: 0000-0003-0112-1624
Pedagogical University of Krakow

Iconic and Symbolic Representation in Early Mathematics Teaching

Reprezentacje ikoniczne i symboliczne w początkowym nauczaniu matematyki

KEYWORDS

early childhood education, mathematics methodology, symbolic representation, iconic representation

ABSTRACT

Mathematics is an abstract science, it uses a specific symbolic language, a kind of code that is difficult for children to comprehend. Using operation signs and numbers is not simple and obvious for them. The learning of mathematics by children cannot therefore be based on symbols alone. Action is needed first. However, since there is a large gap between an action and a symbol, it is necessary to support the teaching of mathematics with graphic means. Introducing a graphic element as a kind of methodological aid requires careful preparation, motivation and embedding it in previous physical activities performed by children. Otherwise, such an item, instead of facilitating the understanding of a concept, hinders that understanding and becomes an illusory aid. This paper describes the role and importance of pictorial representation in early mathematics teaching. Based on selected examples of tasks from textbooks, the difficulties of illustrating mathematical concepts are presented, and appropriate methodological solutions in this area are indicated.

SŁOWA KLUCZE

edukacja wczesnoszkolna, metodyka matematyki, reprezentacja symboliczna, ikoniczna

ABSTRAKT

Matematyka jest nauką abstrakcyjną, posługuje się specyficznym, symbolicznym językiem, rodzajem kodu, który jest trudny dla dzieci. Używanie znaków działań, cyfr nie jest dla nich proste i oczywiste. Uczenie się matematyki przez dzieci nie może więc się opierać jedynie na symbolach. Konieczne jest najpierw działanie. Ponieważ jednak pomiędzy działaniem a symbolem istnieje duża przepaść, to ucząc matematyki należy się posiłkować środkami graficznymi. Wprowadzenie środka graficznego jako swoistego ułatwienia metodycznego wymaga starannego przygotowania, umotywowania oraz osadzenia we wcześniejszych czynnościach fizycznych wykonywanych przez dziecko. W innym wypadku środek taki, zamiast ułatwić zrozumienie jakiegoś pojęcia, utrudnia to rozumienie i staje się ułatwieniem pozornym. W artykule opisano rolę i znaczenie schematów graficznych w początkowym nauczaniu matematyki. Bazując na wybranych przykładach zadań z podręczników, przedstawiono trudności związane z ilustrowaniem pojęć matematycznych oraz wskazano właściwe rozwiązania metodyczne w tym zakresie.

Introduction

Mathematics as an abstract science presents a specific way of looking at the world, and its hermetic language allows us to explain this world in an accurate, precise and unambiguous way. As Ewa Swoboda writes (2017: 27), "it is seen as a science that uses abstract concepts and relationships." And, as the cited author further notes,

> there is no way to experience with the senses what a particular detailed mathematical concept is, or what the relationships and relations between concepts are. These objects and relationships can only be represented through word, image, symbol, gesture. Each such representation carries a specific content, is a code for a certain meaning (Swoboda 2017: 27).

The Specificity of Math Education at the Early Childhood Stage

The specific characteristics of mathematics mean that it cannot be learned simply by observing the actions of others. In mathematics, individual experience is important but also cooperation and exchange of thoughts. It is also important to make mistakes, which are a natural phenomenon and should be skillfully used in the learning process (Turnau 1990: 79). And while this process is inherently based on reasoning and less on experience, the latter plays an important role in the school teaching of mathematics. It becomes particularly important in the case of elementary mathematics, in which concrete manipulation is the starting point in the formation of basic concepts. In teaching mathematics, logical-mathematical experiences are the beginning of a path that leads to higher-level reasoning. Collecting them allows qualitative changes in thinking and the formation of abstract concepts to take place.

Although mathematics is an abstract science that uses a specific symbolic language, teaching this subject to young children cannot be based on symbols alone. The intellectual capacity of early childhood education students limits the use of symbols and therefore makes it necessary to support teaching with practical (physical) actions and pictures. For what is available to adults is not available to 7- to 10-year-old children, and this must be taken into account in their math education. These issues are highlighted by Milan Hejny (1997: 17–18), who describes the cognitive mechanism related to the acquisition of mathematical knowledge. The emergence of a new piece of knowledge is conditioned by motivation and accumulated experiences. These experiences are isolated events at the beginning. Only then, when the child begins to notice the relationships between them, arranges and hierarchizes them, they become universal models. Then a new fragment of knowledge is discovered and this knowledge is understood on a higher, abstract level (Hejny 1997: 17).

In the early stages of children's education, action is the most understandable, because logical thinking, which becomes active at the turn of preschool and early school age, is firmly rooted in action. A shift away from visual perception to purely mental operations takes place later. The students are able to make certain transformations in their minds to conclude, for example, that despite the rearrangement, the lengths of the objects do not change. Action is therefore the beginning of the formation of abstract concepts that are presented symbolically in mathematics.

Since there is a large gap between action (concrete) and symbol (which for some students may be a difficult barrier to overcome), teaching mathematics should be supported by graphical means. Pictorial representations, as Zbigniew Semadeni points out, are

> an intermediate stage—between the concrete and the abstract … which can make it easier for a child to internalize, understand or assimilate certain mathematical notions. A schematic image can serve as a generalization of a specific situation and at the same time a generalization of a purely verbal formulation (Semadeni 1992: 116).

As the author further notes, the introduction of a graphic element as a kind of methodological aid, a bridge between action and symbol, requires very careful preparation, motivation, and embedding in previous physical activities performed by the child. However, if the graphic medium is to be detached from such experiences and introduced as “a finished, static creation, imposed on the child” (Semadeni 1992: 116), it is better not to use it, because instead of making the understanding easier, it will make it more difficult, it will become an illusory aid, a “didactic trick” that brings nothing (Pisarski 1996).

Jerome Seymour Bruner’s Representation Theory as a Learning Model

Reflecting on the role of iconic and symbolic representation in teaching mathematics, it is impossible not to refer to the concept of Jerome Seymour Bruner. According to him, the process of internalization begins with a concrete action, which is summarized in the form of synthetic images to be finally described in the language of symbols. Burner links this transformation of concrete activities into abstract ones, through acting, drawing and telling, to three modes of representation: enactive, iconic, and symbolic defined as “a set of rules in terms of which an individual forms a concept of the constancy of the events he or she has encountered” (Bruner 1978: 530–531). The process of learning, according to Bruner, consists in “creating more economical or efficient ways of representing similar events … in a kind of translation of one mode of representation into another” (Bruner 1978: 531). These modes of representation appear in the child’s life in a well-defined sequence (enactive, iconic, and symbolic), their development is interdependent and each remains virtually intact throughout life.

In younger students’ learning of mathematics, performing an action using manipulatives plays an important role. Action, according to Bruner’s theory, is an enactive representation and is the “representation of past events through an appropriate motor response” (Bruner 1978: 548), it is also the “knowledge of something contained in doing it” (Bruner 1978: 532). Iconic representation, on the other hand, is the knowledge contained in images and pictures. The rules that make up this mode, although they make it possible to grasp an object as a summary image, do not define it in a complete way, but only constitute a summary of what we know about it, and therefore present it approximately, selectively (Bruner 1978: 548). Symbolic representation means coding and decoding using a well-defined character system. Anything that makes up the essence of an individual’s experience can be coded. The basic feature of symbolic representation is its detachment from the concrete, which means that we cannot do with it what we are able to do with a picture or a scheme of action, for the “symbolic system represents things through model features in a distant and arbitrary way. The word neither points directly, here and now, to its referent nor resembles it as an image” (Bruner 1978: 548).

Bruner's modes of representation underpin individual development and define a different way of dealing with incoming information. Each representation contains numerous variations because each represents an event selectively. What is important in this respect is the purpose which the representation is to serve (Bruner 1978: 530–531).

Bruner's learning model finds its application in the early years of learning mathematics. A student entering school is not only expected to be able to function at three levels of representation, but "must be able to move freely from one level of representation to another" (Gruszczyk-Kolczyńska 1994: 85) which means, according to the cited author the "ability to establish relationships between one's actions, graphic representation of things and events, and symbolic representation of them" (Gruszczyk-Kolczyńska 1994: 85). Not being able to function at an appropriate level in this area dooms some children to school failure at the very beginning of their education.

Graphic Representation of Arithmetic Operations in Early Childhood

For many years, various scientific studies in the field of early childhood math education have drawn attention to the various limitations and difficulties of children in understanding the meaning of more or less schematic illustrations or symbolic notations (Gruszczyk-Kolczyńska 1994: 83–102; Nawolska, Żądło 2010: 86–91; 2017: 108–120). With these arguments in mind, it is worth looking at some of the iconic and symbolic representations and their various combinations used in textbook problems for first- to third-grade students to see which of them actually help shape mathematical concepts and which are only an illusory aid (Pisarski 1996).

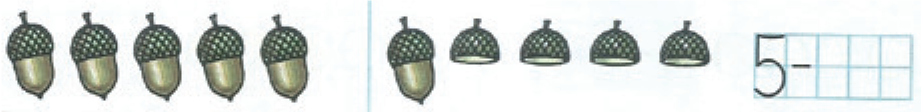
Most problems contained in early childhood education textbooks are supported by a variety of illustrations. At first, they are literal, but later, as children's experiences grow, they become more simplistic and schematic. Each illustration forming part of a problem (arithmetic or verbal) should support thinking, act on children's imagination so that the situation synthesized in a mathematical formula or described in a mathematical text is clear. Such an illustration in the case of arithmetic problems should help to understand the symbolic notation and be helpful in determining the result.

To illustrate children's potential difficulties in interpreting pictures, we will take a look at three selections that illustrate subtraction in the first grade (Figure 1, 2, 3).

Figure 1. Illustrate subtraction

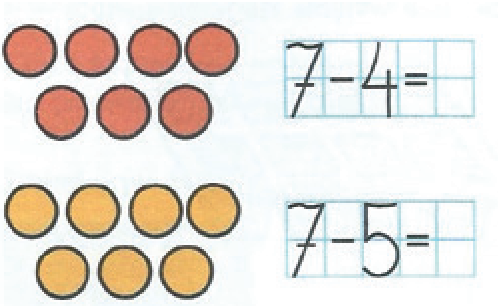
Source: Semadeni (2001: 54).

Figure 2. Illustrate subtraction



Source: Semadeni (2001: 55).

Figure 3. Illustrate subtraction

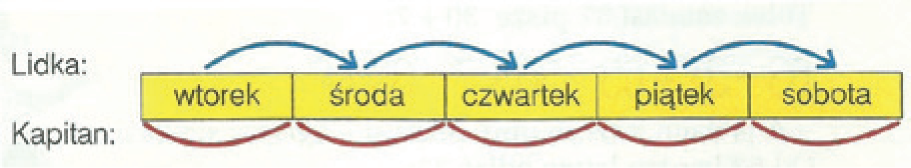


Source: Semadeni (2001: 70).

Subtraction is the inverse operation to addition and for that reason at least it is more difficult. It requires reversal action, which is not easy for students who do not yet reason operationally. In addition, it is difficult to convey the dynamics of the situation and the succession of time in the picture. This is well illustrated in Figure 1. This picture is preceded in the textbook by the *Make up a story. Create a question and answer it* instruction. There is also a blank space to write down the operation. To do this, one must first understand the convention of this static image and decode it according to the accepted way of illustrating subtraction (depletion as a result of: eating, smashing, pouring, ripping, etc.). An additional difficulty is the notation of the operation (7 – 2 = 5), which implies the need to return to the starting point and to imagine seven whole, unbroken cups. The next step is to imagine a change, i.e., breaking two cups. The process does not end there, as the result still needs to be determined. To do this, one needs to skip the subtrahend (2 broken cups) in the picture and focus on the difference, which is five whole cups. A slightly easier (cartoon-like) illustration of subtraction, is presented in Figure 2. There are two images here. The first shows the initial state (there were 5 acorns), while the second shows the final state (1 acorn left). The questions given serve as an aid to reasoning: *How many were there? How many are gone? How many are left?* Admittedly, this time one can see the initial number of acorns, but one still has to visualize the change based on the four acorn cups left. As in the problem in Figure 1, to indicate the difference, one has to omit the subtrahend (4 acorn cups left) and focus on the 1 whole acorn. The difficulty connecting these two illustrations of subtraction is that in both the first and the second picture the number of elements in the set does not change (there were 7 and there are 7, there were 5 and there are 5). Thus, to be able to correctly determine the result of subtraction in these tasks, it is necessary to understand the conventions of the picture, decode the information contained therein (broken, so they are not there, incomplete acorns so they are not there either). The subtraction presented in these two examples, and the difficulties that may arise in understanding the nature of the operation thus presented, justify the need for more frequent recourse to enactive representation in forming the concept of difference. Moving two broken cups (or sticks) out of the 7 cups (or sticks) placed on the desk will then cause no difficulty in determining the result. This is something children in kindergartens can handle. Moving too quickly (especially in case of subtraction, later also of division) to the iconic-symbolic level may not make an operation any easier or more familiar, but more difficult for children to comprehend. In situations such as those in Figures 1 and 2, students often do not see subtraction, and when asked to create stories on their own, these are often such stories as: *There were 5 whole cups and 2 broken cups. How many cups were there?* The next example (Figure 3) is a good illustration of subtraction, provided we properly prepare children to understand it. Such pictures can be introduced after a prior practical action. In order to determine e.g., how many discs will be left when 4 out of 7 are taken away, students should first manipulate (move) such discs, and then write down arithmetic formulas matching such practical activities. Then a picture in which the pushing operation is coded as crossing out can be used. The order of actions to be performed, this time on the picture, will be as follows: first I count the discs, then out of the 7 discs in the picture I separate 4 (because this is how many we are to subtract 7 – 4 =). The remaining discs that are not crossed out constitute the difference. In this case, we can hope that by creating an illustration to the action on their own, the students will remember the successive stages of that creation and the sequence of time. The picture, therefore, should be comprehensible, despite the fact that it is still static in the end, which is a fundamental difficulty in illustrating subtraction (Nawolska, Żądło-Treder 2017).

The use of illustrations in word problems is intended to help visualize the situation described in the problem and facilitate understanding and correct solution. A good example of such an illustration is Figure 4 corresponding to the following problem: *The ship entered the port on Tuesday before dawn, when it was still dark. It will depart on Saturday just before midnight. Help the captain calculate how many days the ship will be in the port.*

Figure 4. Illustration of the word problem

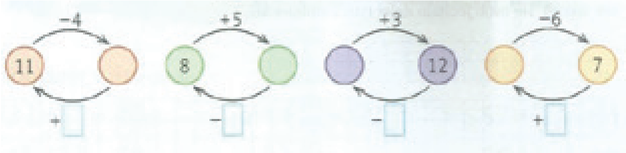


Source: Lankiewicz, Semadeni (1994: 78).

This word problem is about calendar calculations, specifically knowing the days of the week. It also checks the understanding of the term "day," which is not unambiguous and can be interpreted in different ways by children; it also refers to the problem of counting in "including" and "excluding" terms. The calculation of Lidka and the Captain shown in the illustration shows two possible methods. The Captain, reasoning practically (I entered before dawn on Tuesday and leave before midnight on Saturday), counts consecutive days in calendar terms. He will therefore spend 5 days in the port. Lidka, on the other hand, treats a day as a 24-hour segment and counts the passage of time, in which one day passes from Tuesday to Wednesday, another from Wednesday to Thursday, and so on until Saturday. Hence, according to her, the Captain will spend 4 days in the port. This apparent contradiction is very well explained by the proposed pictorial representation. However, this is not always the case. It happens that the pictures accompanying the problem serve a decorative purpose only and do not contribute anything significant to the process of solving them (Swoboda 2017: 36). These types of situations can even block one's understanding of the problem and make it more difficult, not easier, to solve.

In the math education of children, it is very important to discover general mechanisms on the basis of generalized ways of doing things, while it is a mistake to draw general conclusions on the basis of single, isolated facts. Often the pictorial representations presented to children in textbooks (trees, function tables, graphs), which are mixed iconic-symbolic creations, constitute ready products, introduced without any previous preparation. Students who are not adequately prepared to decode information contained in pictorial representations can rightly feel confused because they may not understand a problem coded in such a manner. It is therefore doubtful that at further stages of education the essence of the representation can be understood and used correctly (Gruszczyk-Kolczyńska 1994; Nawolska, Żądło 2010, 2017; Semadeni 1992). An example of a problem using an arrow graph to illustrate the relationship between addition and subtraction is shown in Figure 5.

Figure 5. Arrow graph

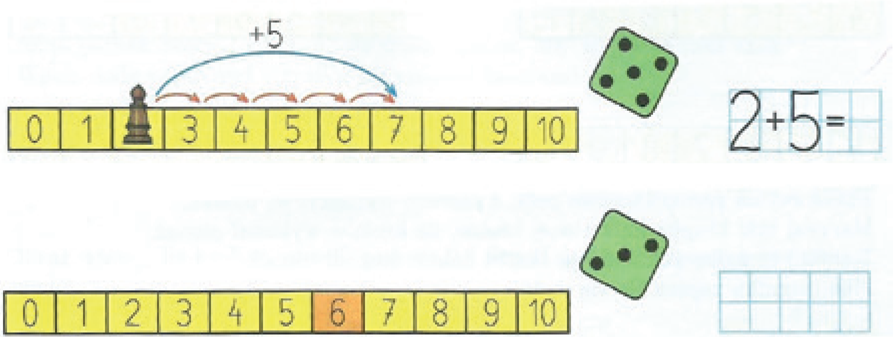


Source: Dobrowolska, Jucewicz, Szulc (2014: 49).

This problem, like most such problems in textbooks, is not preceded by any commentary, but only contains the *Complete the graphs* instruction. Unless the teacher comments on such a task, students will most often automatically fill in the missing numbers. They do so without giving any further thought to the meaning of these notations.

However, other examples exist in textbooks. Figures 6, 7, 8, 9, 10 illustrate the steps of introducing an arrow graph.

Figure 6. Step 1

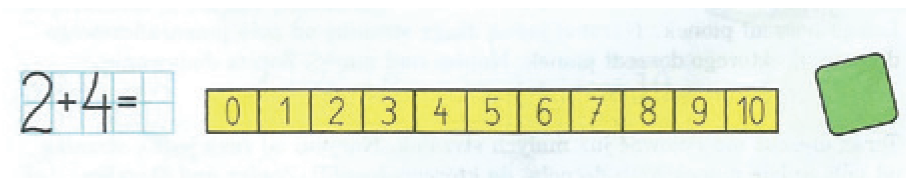


Source: Semadeni (2001: 37).

Figure 7. Step 2

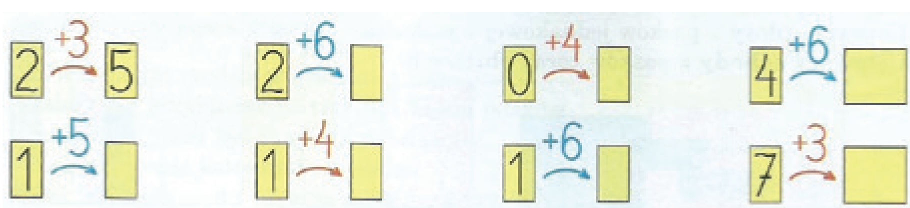
Source: Semadeni (2001: 37).

Figure 8. Step 3



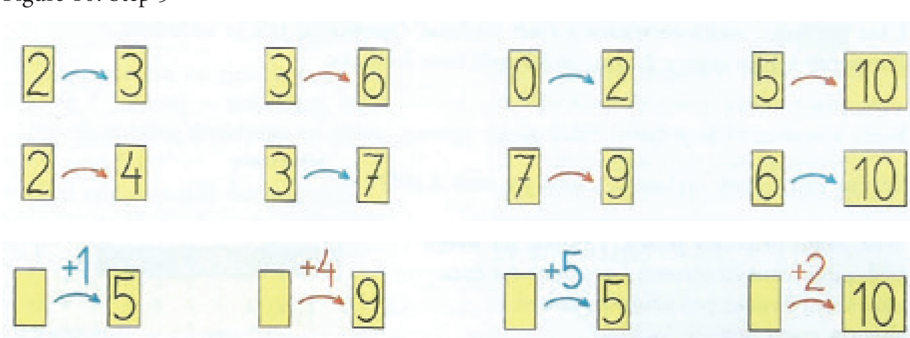
Source: Semadeni (2001: 38).

Figure 9. Step 4



Source: Semadeni (2001: 39).

Figure 10. Step 5



Source: Semadeni (2001: 39).

It all begins with a race game (using a row of squares with numbers) in which students play by rolling the dice and moving their pawns one square for each dot on the dice (Semadeni 1992: 114–119). At this stage, they can already begin to code the movements of their pawns along the row by drawing the appropriate arrows. In the next stage, this real-life game is transferred to the pages of the textbook. Students deal with hypothetical games played by the characters in a given problem. They should have no difficulties in identifying with them because they have played such games in the classroom. Thus, in Figure 6, they move one square to the right for each spot on the dice. In doing so, they first draw small arrows and complete the operation notation. Through direct experience, the formal notation 2 + 5 = 7 is clear and not difficult to relate to the pictorial representation. The number 2 is where the pawn stood, 5 is the number of dots thrown, and the plus sign and right arrow indicate the direction of the pawn's movement (right, add). The number 7 is the symbol of the square the pawn landed on after moving to the right. The tasks in Figures 7 and 8 deepen the understanding of this code (numbers, operation signs). The children's task is to determine, based on the illustration (Figure 7) or the written arithmetic formula (Figure 8), the number of dots thrown, the start and finish square. The problems presented in Figure 9 and Figure 10 contain a typical single-operation arrow graph. After such thorough preparation beforehand, it can hardly be called something imposed in advance, introduced without proper motivation. It provides an abbreviated record of the pawn's journey on the board. We have the square on which the pawn was standing, the arrow indicating the direction of the pawn's movement with the number of dots thrown and the square to which the pawn was moved. At this point, such a pictorial representation should not be something incomprehensible to children, for as Margaret Donaldson noted long ago "all normal children are able to demonstrate their abilities as thinking beings and users of language (...) provided, however, that they are confronted with meaningful life situations" (Donaldson 1986: 161).

The examples from the textbook show that the language of mathematics differs significantly from natural language. It is a difficult language, and as a result, children's use of pictorial representations or symbols such as operation signs, numbers, or arithmetic formula notation is not as obvious. Every single mathematical symbol (+, –, ·, :) that a child learns at an early stage of education is not only an abbreviated notation of a mathematical operation, but also a coded action, expressed with an appropriate verb (combine/add, take away/subtract, take "a few several times," "distribute equally," etc.). When we place these symbols between numbers, a mathematical formula is formed, which, again, is not a single symbolic notation. For a child who is just gathering logical-mathematical experiences in the course of various activities, the formula notation should be a synthesis of various experiences. For example, 10 : 2 = 5 is not only a notation of the relationship between the numbers 10, 2, and 5, nor only a notation of the specific quotient of the two numbers, but first of all an abbreviated (symbolic) notation of the action of dividing, for example, 10 candies into 2 equal portions and determining how many of them will be in such a portion, as well as a symbolic notation of dividing 10 candies into portions of 2 candies each and determining how many such portions will be formed. Furthermore, dividing ten candies into two equal portions is not the same for a child as pouring 10 liters of water into two buckets equally, or pouring 10 liters of water into vessels, two liters into each. The number 5 which is the solution to these two different problems represents in the first case the amount of water in one vessel, and in the next case the number of vessels needed. Formula 10 : 2 = 5 is a synthesis of the action of dividing relating to these different life situations (dividing candies, pouring water, and many other ones not mentioned here). This simple example shows that such a single symbolic notation contains a variety of children's experiences. Thus, it is difficult to expect such a notation to be immediately legible to all children because, as mentioned earlier, it represents a synthesis of diverse, sometimes extremely different children's experiences. On the other hand, however, it is precisely this variety of experiences (activities, actions), children's personal activity in this area, independence in constructing knowledge that contributes to increasing the operability of language and its construction at an ever-higher level. Hence, the introduction of symbolic notations (symbolic language) must be properly motivated and linked to the personal activity of the learner. Thus, it would be good for early childhood education teachers to be aware that for a certain group of children entering school, the symbolic level remains inaccessible for a long time (Gruszczyk-Kolczyńska 1994: 83–102). For them, the formal notation of a mathematical operation is sometimes incomprehensible, although they can answer how many candies each child will receive, or how candies each child will have (how much water will be in one container, and how many containers are needed). However, they are not yet aware that it is always the case that 10 : 2 = 5, regardless of whether we are pouring water or dividing candies.

To conclude the discussion of the language of mathematics as a kind of symbolic code, it is worth noting that in addition to the language of textbook problems, students still have to deal with the more or less formal language of the teacher. This language is also a code and it is worthwhile for a teacher to use it in a thoughtful way that allows children to understand it. The lack of ability to decode incoming information may cause disturbances in communication between the student and the teacher

> … distortions or disruptions may arise as a result of the student's inadequate decoding of the received information (the student creates a wrong idea about situations, processes, objects) and its translation into the internal language or vice versa—the lack of ability

to code their own thoughts and translate information from their own language into the language in which the student communicates with the teacher (Bugajska-Jaszczołt, Czajkowska 2013: 40).

Conclusion

The basic condition for limiting the spread of the phenomenon of the so-called "degenerate formalism"[1] is to create appropriate conditions for students to learn mathematics and properly develop their mathematical activity. Intellectual activity should take place on many levels and be based on student independence in pursuit of knowledge. The use of symbolic, mathematical language in this area is not straightforward because "like any sign, a mathematical symbol carries a specific meaning. It should be associated with various procedures, with typical uses. However, such associations are built over a long period of time, through the accumulation of experiences" (Swoboda 2017: 37). Good illustrations, pictorial representations properly constructed and grounded in children's experience can be helpful in building operational knowledge. Knowledge that will be owned by the learner, not something imposed and completely incomprehensible. In doing so, both the action and the illustration must not only precede the symbols, but also accompany them. The same is true of language. The symbolic one cannot be imposed on children too early, it should be preceded and supported by the colloquial one (Dąbrowski 2008: 142–143). Language skills, related to translating natural language into the symbolic language of mathematics, play an important role in interpreting mathematical problems. To be able to make such a translation one needs knowledge and understanding of the relationship between the two languages: natural and symbolic (Rodríguez-Hernández, Pruneda, Rodríguez-Díaz 2021: 3). It is important to remember that everyday language and the language of mathematics are two different worlds. Thus, for example, a circle that exists in the real world is just a shape of a real object, e.g., of a wheel in a bicycle or a car. In mathematics, a circle is an abstract concept, a product of our mind, defined in theory and not existing in the real world. The students in the lesson can at most use its model. However, when teachers use the term "circle," they do not mean a shape of a real object but an abstract object (Bugajska-Jaszczołt, Czajkowska 2013: 40).

Breaking with the "concrete" and realizing certain mathematical regularities (e.g., independence of numbers and numerical operations from the real world)

[1] The term introduced by Zofia Krygowska (in Polish: *formalizm zdegenerowany*). A manifestation of degenerate formalism is a lack of understanding of the meaning of terms, mathematical symbols, or chaos in the application of syntactic rules (Krygowska 1986: 27).

is a crucial moment in the development of mathematical knowledge. This moment, called by Hejny the elevation of abstraction (Hejny 1997: 17–18), allows the child to understand math at a higher level. It begins to understand symbols, rules, laws, relationships, procedures, etc. For this elevation of abstraction to take place and for a child to gain access to the abstract world of mathematics, prior experience is needed at all levels of representation—enactive, iconic, and symbolic according to Brunner's theory—and this is the basis for functioning well in the school setting.

Bibliography

Bruner J.S. (1978). *Poza dostarczone informacje. Studia z psychologii poznawania*, trans. B. Mroziak, Warszawa: Państwowe Wydawnictwo Naukowe.

Bugajska-Jaszczołt B., Czajkowska M. (2013). *Komunikacja na zajęciach z edukacji matematycznej*, “Problemy Wczesnej Edukacji”, vol. 4, no. 4(23), pp. 39–50.

Dąbrowski M. (2008). *Pozwólmy dzieciom myśleć. O umiejętnościach matematycznych polskich trzecioklasistów*, Warszawa: Centralna Komisja Egzaminacyjna.

Dobrowolska M., Jucewicz M., Szulc A. (2014). *Lokomotywa. Klasa III. Z. 1*, Gdańsk: Gdańskie Wydawnictwo Oświatowe.

Donaldson M. (1986). *Myślenie dzieci*, trans. A. Hunca-Bednarska, E.M. Hunca, Warszawa: Wiedza Powszechna.

Gruszczyk-Kolczyńska E. (1994). *Dzieci ze specyficznymi trudnościami w uczeniu się matematyki*, Warszawa: Wydawnictwa Szkolne i Pedagogiczne.

Hejny M. (1997). *Rozwój wiedzy matematycznej*, “Dydaktyka Matematyki”, no. 19, pp. 15–28.

Krygowska Z. (1986). *Elementy aktywności matematycznej, które powinny odgrywać znaczącą rolę w matematyce dla wszystkich*, “Dydaktyka Matematyki”, no. 6, pp. 25–41.

Lankiewicz B., Semadeni Z. (1994). *Matematyka 2. Klasa 2*, Warszawa: Wydawnictwa Szkolne i Pedagogiczne.

Nawolska B., Żądło J. (2010). *Błąd w edukacji matematycznej*, [in:] K. Gąsiorek, Z. Nowak (eds.), *Tworzenie obrazu świata u dzieci w młodszym wieku szkolnym. Szanse i bariery*, Kraków: Wydawnictwo Naukowe Uniwersytetu Pedagogicznego, pp. 78–94.

Nawolska B., Żądło-Treder J. (2017). *Dziecięca koncepcja matematyki tworzona przez zadania z podręczników szkolonych*, [in:] I. Czaja-Chudyba, B. Pawlak, J. Vaškevič-Buś (eds.), *Wizja świata – wizja dziecka w przestrzeni podręczników do edukacji wczesnoszkolnej*, Kraków: Wydawnictwo Naukowe Uniwersytetu Pedagogicznego, pp. 108–120.

Pisarski M. (1996). *Ułatwienia metodyczne w nauczaniu początkowym matematyki*, “Problemy Studiów Nauczycielskich. Dydaktyka Matematyki”, no. 6, pp. 144–158.

Semadeni Z. (1992). *Grafy strzałkowe i drzewa jako reprezentacje ikoniczno-enaktywne*, “Dydaktyka Matematyki”, no. 14, pp. 114–119.

Semadeni Z. (2001). *Słoneczna matematyka. Klasa 1. Zeszyt jesienny*, Warszawa: Muza Szkolna.

Swoboda E. (2017). *Tworzenie różnych reprezentacji przez dzieci podczas rozwiązywania problemu matematycznego*, "Edukacja", no. 1(140), pp. 27–38. DOI: 10.24131/3724.170102.

Turnau S. (1990). *Wykłady o nauczaniu matematyki*, Warszawa: Państwowe Wydawnictwo Naukowe.

Netography

Rodríguez-Hernández M.M., Pruneda R.E., Rodríguez-Díaz J.M. (2021). *Statistical analysis of the evolutive effects of language development in the resolution of mathematical problems in primary school education*, "Mathematics", 2021, vol. 9, no. 1081, pp. 1–14, https://doi.org/10.3390/math9101081 (accessed: 5.07.2021).

ADDRESS FOR CORRESPONDENCE

Joanna Żądło-Treder
Pedagogical University of Krakow
e-mail: joanna.zadlo-treder@up.krakow.pl